

FREDHOLM PROPERTIES OF SCHRÖDINGER OPERATORS IN $L^p(\mathbb{R}^N)$

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Abstract. We consider real potentials V such that the Schrödinger operator $-\Delta + V$ maps the Sobolev space $W^{2,p}(\mathbb{R}^N)$ continuously into $L^p(\mathbb{R}^N)$ for a range of values of p which includes 2. Let σ_e denote the essential spectrum of $-\Delta + V$ as a self-adjoint operator in $L^2(\mathbb{R}^N)$. If $\lambda \notin \sigma_e$, we show that for all p in the range considered, $-\Delta + V - \lambda : W^{2,p}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ is a Fredholm operator of index zero, that $\ker \{-\Delta + V - \lambda\}$ is independent of p and that $L^p(\mathbb{R}^N) = \ker \{-\Delta + V - \lambda\} \oplus \{-\Delta + V - \lambda\}W^{2,p}(\mathbb{R}^N)$.

1. Introduction. Under appropriate conditions on the potential V , the Schrödinger operator $-\Delta + V$ defines a self-adjoint operator $S : D(S) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ with domain $D(S)$ equal to the usual Sobolev space $H^2(\mathbb{R}^N) = W^{2,2}(\mathbb{R}^N)$. Let us denote the spectrum of S by σ , and its essential spectrum by σ_e . The discrete spectrum, σ_d , of S is the set of isolated points in σ which are eigenvalues of finite multiplicity. Recall (see [3], [22], [20] for these notions) that $\sigma = \{\lambda \in \mathbb{R} : S - \lambda I : W^{2,2}(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N) \text{ is not an isomorphism}\}$ and that $\sigma_d = \sigma \setminus \sigma_e = \{\lambda \in \sigma : S - \lambda I : W^{2,2}(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N) \text{ is a Fredholm operator}\}$. For $\lambda \in \sigma_d$, this means that

(i) $\ker(S - \lambda I) = \{u \in W^{2,2}(\mathbb{R}^N) : (S - \lambda I)u = 0\} \neq \{0\}$ but has finite dimension,

(ii) $\text{rge}(S - \lambda I) = (S - \lambda I)W^{2,2}(\mathbb{R}^N)$ is a closed subspace of $L^2(\mathbb{R}^N)$, and

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$$(iii) \ L^2(\mathbb{R}^N) = \ker(S - \lambda I) \oplus \operatorname{rge}(S - \lambda I).$$

We observe that the properties (i) and (ii) are simply a restatement of the fact that $-\Delta + V - \lambda : W^{2,2}(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is a Fredholm operator. The property (iii) follows from (i), (ii) and the self-adjointness of S . It implies that the index of $S - \lambda I$ is zero and that the geometric and algebraic multiplicities of the eigenvalue λ are equal.

Our purpose in this note is to show that these properties also hold when $-\Delta + V$ is considered as an operator from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for a whole range of values of p . As a special case of our main result we obtain the following information.

Theorem 1. *Let $V \in L^\infty(\mathbb{R}^N)$. Then $-\Delta + V : W^{2,2}(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is a self-adjoint operator whose spectrum and discrete spectrum are denoted by σ and σ_d respectively. For $p \in (1, \infty)$, consider also the operator $S_p : W^{2,p}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ defined by*

$$S_p u = (-\Delta + V)u \quad \text{for } u \in W^{2,p}(\mathbb{R}^N).$$

For every $p \in (1, \infty)$, the following conclusions are valid.

- (i) $S_p - \lambda I : W^{2,p}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ is an isomorphism if $\lambda \notin \sigma$, whereas, if $\lambda \in \sigma_d$, then*
- (ii) $S_p - \lambda I : W^{2,p}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ is a Fredholm operator of index zero,*
- (iii) $\ker(S_p - \lambda I) = \ker(S_2 - \lambda I)$, and*
- (iv) $L^p(\mathbb{R}^N) = \ker(S_p - \lambda I) \oplus \operatorname{rge}(S_p - \lambda I)$ where \oplus denotes a topological direct sum.*

Our interest in these properties of S_p stems from work in bifurcation theory ([10], [12] and [24]) where $S_p - \lambda I$ arises as the linearization of a nonlinear partial differential operator. In dealing with nonlinear partial differential operators it is often important to work in Sobolev spaces with $p > 2$ in order to avoid imposing undesirable restrictions on the growth of the nonlinear terms. If $\lambda \in \sigma_d$, then parts (ii) and (iii) of Theorem 1 show that $S_p - \lambda I : W^{2,p}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ is a Fredholm operator of index zero and $\ker(S_p - \lambda I) = \ker(S_2 - \lambda I)$. But since λ is an isolated point of σ , part (i) ensures that there exists $\varepsilon > 0$ such that $S_p - \mu I : W^{2,p}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N)$ is an isomorphism for all $\mu \in [\lambda - \varepsilon, \lambda) \cup (\lambda, \lambda + \varepsilon]$. Finally, the geometric and algebraic multiplicities of λ , as an eigenvalue of S_p , are equal. These properties are very useful in various contexts in nonlinear analysis such as Lyapunov-Schmidt reduction ([15], [19]) or degree theoretic calculations ([1], [4], [13]) where they imply significant simplifications.

The conclusions of Theorem 1 were established in [10] under the additional assumption that $\lambda < \liminf_{|x| \rightarrow \infty} V(x)$ which is required there because the maximum principle is used in an essential way on a neighbourhood of infinity. That approach was extended in [12] to cover singular potentials, the results being established for S_p provided that $V \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ where $p > \frac{N}{2}$ and $\lambda < \liminf_{|x| \rightarrow \infty} V(x)$. In this paper we use a new approach based on a perturbation result established in Section 3. It enables us to deal with all $\lambda \in \sigma_d$, whereas the afore mentioned results were restricted to the region $(-\infty, \liminf_{|x| \rightarrow \infty} V(x))$ which always lies below the essential spectrum of S_2 . We also allow singular potentials and the results in Section 2 establish some basic facts required for this. The main result, of which Theorem 1 is a special case, is given in Section 4.

The conclusions of Theorem 1 and its generalization Theorem 9 show that some spectral properties of the Schrödinger operator acting on the space $L^p(\mathbb{R}^N)$ are in fact independent of the choice of p . The fact that the spectrum, as a set, does not depend on p has been established by various approaches and for a broad class of potentials, [16], [17]. Our work differs from these contributions in two fundamental aspects. First of all, it deals with the p -independence of some finer properties of points in the spectrum and, secondly, the domain of the operator in $L^p(\mathbb{R}^N)$ is required to be the space $W^{2,p}(\mathbb{R}^N)$ rather than simply making a statement about some realization of $-\Delta + V$ in $L^p(\mathbb{R}^N)$ whose domain need not be $W^{2,p}(\mathbb{R}^N)$. In much of the work concerning the L^p -theory of $-\Delta + V$, the realization of this operator is taken to be the generator of the Schrödinger semigroup on $L^p(\mathbb{R}^N)$. It is not a trivial matter to prove that, for a given class of potentials, the domain of this generator is precisely $W^{2,p}(\mathbb{R}^N)$. Indeed this has been made possible only recently by the work of Ishikawa, [25]. Our approach makes no appeal to semigroup theory. Instead we rely on an integral representation of the solutions and on known estimates for the kernel, together with some standard results from spectral theory.

2. Preliminary results. Our first objective is to introduce a class of potentials V (allowing singularities) for which the operator $-\Delta + V$ is an unbounded self-adjoint operator on $L^2(\mathbb{R}^N)$ with domain $W^{2,2}(\mathbb{R}^N)$ and a bounded operator from $W^{2,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ for some range of values of p including 2. The following condition defines one such class which is frequently used in mathematical physics. We deal only with real-valued functions.

(H)_q There is some $q \geq 2$ with $q > \frac{N}{2}$ such that the potential $V = P + Q$

where $P \in L^\infty(\mathbb{R}^N)$ and $Q \in L^r(\mathbb{R}^N)$ for all $r \in [1, q]$. In this case we write $V \in T_N(q)$.

Remarks. 1) The case $q = \infty$ reduces to setting $Q = 0$. Clearly $L^\infty(\mathbb{R}^N) = T_N(\infty) \subset T_N(q_1) \subset T_N(q)$ for $q < q_1 < \infty$. When we write $V \in T_N(q)$ it is to be understood that $q \geq 2$ with $q > \frac{N}{2}$.

2) For $N = 3$, the Coulomb potential $V(x) = \frac{1}{|x|}$ is in the class $T_N(q)$ for $2 \leq q < 3$, as is easily seen by setting $Q(x) = \chi(x)|x|^{-1}$ and $P(x) = \{1 - \chi(x)\}|x|^{-1}$, where χ is the characteristic function of the unit ball in \mathbb{R}^3 .

3) The potentials satisfying $(H)_q$ are sometimes said to be of Kato-Rellich type, [20]. As is noted below in Theorem 3, $-\Delta + V$ is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $W^{2,2}(\mathbb{R}^N)$ for potentials of this type. In the L^p -theory of Schrödinger operators much broader classes of potentials have been treated (see [16], [17], [5]), but then the domain on which $-\Delta + V$ is self-adjoint in $L^2(\mathbb{R}^N)$ may not be $W^{2,2}(\mathbb{R}^N)$. For example, in the discussion of the L^p -properties of eigenfunctions of $-\Delta + V$, potentials in the class K_N are often used ([5], [9]) and all potentials satisfying the condition $(H)_q$ are of this type. Since we shall make some use of these results (see Proposition 6) we recall the definition of the class K_N for the reader's convenience. A potential V belongs to the (Kato) class K_N provided that

$$\lim_{\alpha \rightarrow 0+} \left\{ \sup_{x \in \mathbb{R}^N} \int_{|x-y| \leq \alpha} \gamma_N(|x-y|) |V(y)| dy \right\} = 0$$

where, for $t > 0$,

$$\gamma_N(t) = \begin{cases} 1 & \text{if } N = 1 \\ -\ln t & \text{if } N = 2 \\ t^{2-N} & \text{if } N \geq 3 \end{cases}.$$

Clearly, $L^q(\mathbb{R}^N) \subset K_N$ if $q \geq 2$ with $q > \frac{N}{2}$, from which it follows that every potential satisfying $(H)_q$ is in the class K_N .

We now begin the study of $-\Delta + V$ as an operator between the spaces $W^{2,p}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$. We simplify the notation by setting

$$W^{2,p}(\mathbb{R}^N) = X_p \quad \text{and} \quad L^p(\mathbb{R}^N) = Y_p,$$

where the usual norms on these spaces are denoted by $\|\cdot\|_{2,p}$ and $|\cdot|_p$ respectively. We recall (see Corollaire IX.13 of [11], for example) that X_p is

continuously embedded in Y_s provided that $1 \leq p \leq \infty$ and

$$\begin{cases} p \leq s \leq \infty & \text{if } p > \frac{N}{2} \\ p \leq s < \infty & \text{if } p = \frac{N}{2} \\ p \leq s \leq p^* & \text{if } p < \frac{N}{2} \end{cases} \quad (1)$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{2}{N}$. Furthermore, there is a constant $C = C(s, p, N)$ such that the following estimate holds (see Corollary 2 in Section 1.4.7 of [23], for example)

$$|u|_s \leq C \|u\|_{2,p}^\tau |u|_p^{1-\tau} \quad (2)$$

provided that s satisfies (1) where $\tau = \frac{N}{2} \left(\frac{1}{p} - \frac{1}{s} \right)$ if $s < \infty$ and $\tau = \frac{N}{2p}$ if $s = \infty$. Note that $1 - \tau > 0$ in all cases except for $s = p^*$ when $p < \frac{N}{2}$.

Lemma 2. *Suppose that $Q \in L^r(\mathbb{R}^N)$ for all $r \in [1, q]$ for some q such that $\frac{N}{2} \leq q \leq \infty$. Then $Qu \in Y_p$ for all $u \in Y_s$ and there exists a constant $C(s, p)$ such that*

$$|Qu|_p \leq C(s, p) |u|_s \quad \text{for all } u \in Y_s$$

provided that $p \leq q$ and

$$\begin{cases} \frac{1}{\frac{1}{p} - \frac{1}{q}} \leq s \leq \infty & \text{if } 1 \leq p < q \\ s = \infty & \text{if } p = q \end{cases}. \quad (3)$$

Proof. Suppose first that $1 \leq p \leq q$ and $s = \infty$. Then

$$|Qu|_p \leq |Q|_p |u|_\infty < \infty.$$

Suppose now that $1 \leq p < q$ and that $\frac{1}{\frac{1}{p} - \frac{1}{q}} \leq s < \infty$. Then $s \geq p$ and, by Hölder's inequality,

$$|Qu|_p \leq |Q|_{pt} |u|_{pt'} = |Q|_{pt} |u|_s$$

where $t' = \frac{s}{p} \in [1, \infty)$ and $\frac{1}{t} + \frac{1}{t'} = 1$. If $t' = 1$, then $q = \infty, p = s$ and $pt = \infty$. If $t' > 1$, then

$$\frac{1}{pt} = \frac{1}{p} - \frac{1}{s} \geq \frac{1}{q} \quad \text{and so } p \leq pt \leq q.$$

Hence $|Q|_{pt} < \infty$ and the result is proved. \square

This leads immediately to the following result.

Theorem 3. *Let the potential $V \in T_N(q)$. Then $-\Delta + V$ is a bounded linear operator from X_p into Y_p which will be denoted henceforth by*

$$S_p : X_p \rightarrow Y_p, \quad (4)$$

provided that $1 \leq p \leq q$. Furthermore, $S_2 : X_2 \subset Y_2 \rightarrow Y_2$ is a self-adjoint operator.

Proof. Clearly $-\Delta + P$ is a bounded linear operator from X_p into Y_p for any $p \in [1, \infty]$ and so it is enough to verify that multiplication by Q takes X_p boundedly into Y_p under the conditions stated above. But, using the continuous embeddings of X_p into Y_s given by (1), this follows from Lemma 2. The fact that $-\Delta + V : X_2 \subset Y_2 \rightarrow Y_2$ is self-adjoint is a well-known consequence of the Kato-Rellich perturbation theorem; see [2], [20], [18] for example. \square

Remark. The self-adjointness of $-\Delta + V : X_2 \subset Y_2 \rightarrow Y_2$ can be established for a larger class of potentials. For example, this is proved in Theorem 10.20 of [22] under the hypothesis (Stummel conditions) that $V \in M_\rho(\mathbb{R}^N)$ for some $\rho < 4$ where $V \in M_\rho(\mathbb{R}^N)$ means that

$$\begin{cases} \sup_{x \in \mathbb{R}^N} \int_{|x-y|} |x-y|^{\rho-N} V(y)^2 dy & \text{if } \rho < N \\ \sup_{x \in \mathbb{R}^N} \int_{|x-y|} V(y)^2 dy & \text{if } \rho \geq N \end{cases}.$$

The hypothesis $(H)_q$ implies that $V \in M_\rho(\mathbb{R}^N)$ for some $\rho < 4$, since in this case $V \in M_\rho(\mathbb{R}^N)$ for $\rho = N$ if $N \leq 3$ and for $\rho > \frac{2N}{q}$ if $N \geq 4$. There is an illuminating comparison of the Stummel and Kato classes in [9].

We now turn to some issues concerning the inverses of these operators. In this discussion the Calderon-Zygmund inequality plays a fundamental role.

We recall that if $u \in L^p(\mathbb{R}^N)$ for some $1 \leq p \leq \infty$, then u defines a tempered distribution.

Lemma 4. *Let h be a tempered distribution on \mathbb{R}^N and consider the equation*

$$-\Delta u + u = h \quad (5)$$

in the sense of distributions.

(a) *There is a unique tempered distribution $u = \Gamma(h)$ satisfying (5).*

(b) If $h \in Y_p$ for some $p \in (1, \infty)$, then $\Gamma(h) \in X_p$ and there exists a constant $C(N, p)$ such that

$$\|\Gamma(h)\|_{2,p} \leq C(N, p) |h|_p \text{ for all } h \in Y_p. \quad (6)$$

(c) For $p \in (1, \infty)$, $(-\Delta + 1) : X_p \rightarrow Y_p$ is an isomorphism.

Proof. For part (a), see the statement 1) of Proposition 27 in Chapter 2 of [2]. Part 2) of the same proposition shows that $\Gamma(h) \in X_p$ with $|\Gamma(h)|_p \leq |h|_p$ when $h \in Y_p$ and $p \in (1, \infty)$. Then (6) follows from the Calderon-Zygmund inequality, see (9.33) in [21], for example. This proves part (b) and part (c) is an immediate consequence of parts (a) and (b). \square

Under the same restrictions as in Theorem 1 we obtain the following estimate.

Lemma 5. Let the potential $V \in T_N(q)$. Then, for each $p \in (1, q] \cap (1, \infty)$, there is a constant $C(p)$ such that

$$\|u\|_{2,p} \leq C(p) \{|u|_p + |S_p u|_p\} \text{ for all } u \in X_p.$$

Proof. Fix a value of p satisfying the above restrictions and suppose that there is no constant such that the conclusion holds. Then there must be a sequence $\{w_n\} \subset X_p$ such that $\|w_n\|_{2,p} = 1$ for all $n \in \mathbb{N}$ whereas $|w_n|_p \rightarrow 0$ and $|S_p w_n|_p \rightarrow 0$. Using the estimate (2) we obtain

$$|w_n|_s \leq C \|w_n\|_{2,p}^\tau |w_n|_p^{1-\tau} = C |w_n|_p^{1-\tau}$$

provided that

$$\begin{cases} p \leq s \leq \infty & \text{if } p > \frac{N}{2} \\ p \leq s < \infty & \text{if } p = \frac{N}{2} \\ p \leq s \leq p^* & \text{if } p < \frac{N}{2} \end{cases} \quad \text{and } \tau = \begin{cases} \frac{N}{2} \left(\frac{1}{p} - \frac{1}{s} \right) & \text{if } s < \infty \\ \frac{N}{2p} & \text{if } s = \infty \end{cases}.$$

Thus,

$$|w_n|_s \rightarrow 0 \text{ for } \begin{cases} p \leq s \leq \infty & \text{if } p > \frac{N}{2} \\ p \leq s < \infty & \text{if } p = \frac{N}{2} \\ p \leq s < p^* & \text{if } p < \frac{N}{2} \end{cases}$$

since $1 - \tau > 0$ in these cases. In particular, Lemma 2 and the restrictions on p ensure that $|Qw_n|_p \rightarrow 0$. Consequently,

$$|(V - 1)w_n|_p = |(P + Q - 1)w_n|_p \rightarrow 0$$

under the same conditions since $P - 1 \in L^\infty(\mathbb{R}^N)$. But, using Lemma 4(b),

$$\begin{aligned} 1 &= \|w_n\|_{2,p} \leq C(N, p) |(-\Delta + 1)w_n|_p \\ &\leq C(N, p) \left\{ |(-\Delta + V)w_n|_p + |(V - 1)w_n|_p \right\}, \end{aligned}$$

which contradicts the fact that $|S_p w_n|_p = |(-\Delta + V)w_n|_p \rightarrow 0$. This proves the result. \square

It will be necessary to use several known results about eigenfunctions of Schrödinger operators and, for easy reference, we formulate them in our context in the following proposition. See [7] for the earliest results of this type due to Schnol, and [9], [5] for related material.

Proposition 6. *Let the potential $V \in T_N(q)$ and consider the operator S_p defined by (4) where $p \in (1, q] \cap (1, \infty)$. Let σ_e denote the essential spectrum of the self-adjoint operator $S_2 : X_2 \subset Y_2 \rightarrow Y_2$. Suppose that $\lambda \in \mathbb{R}$, $u \in X_p$ and $S_p u = \lambda u$.*

(1) *After modification on a set of measure zero,*

$$u \in C(\mathbb{R}^N) \quad \text{and} \quad \lim_{|x| \rightarrow \infty} u(x) = 0.$$

(2) *If $\lambda \notin \sigma_e$ and $u \in Y_\infty$, then $u \in Y_2$.*

(3) *If $\lambda \notin \sigma_e$ and $u \in Y_2$, there exist constants C and $\delta > 0$, such that*

$$|u(x)| \leq C e^{-\delta|x|} \quad \text{for all } x \in \mathbb{R}^N.$$

(4) *If $\lambda \notin \sigma_e$, then $u \in Y_s$ for all $s \in [1, \infty]$.*

Proof. (1) Clearly $u \in L^1_{loc}(\mathbb{R}^N)$ and, by (1) and Lemma 2, it follows that $Vu \in L^1_{loc}(\mathbb{R}^N)$. As we have already noted, V belongs to the Kato class K_N and so the continuity of u is a consequence of Theorem C.1.1 of [5]. Furthermore, by Theorem C.1.2 of [5], there is a constant C (independent of x) such that

$$|u(x)| \leq C \int_{|x-y| \leq 1} |u(y)| dy \quad \text{for all } x \in \mathbb{R}^N.$$

Since

$$\int_{|x-y| \leq 1} |u(y)| dy \leq \left\{ \int_{|x-y| \leq 1} dy \right\}^{\frac{1}{p'}} \left\{ \int_{|x-y| \leq 1} |u(y)|^p dy \right\}^{\frac{1}{p}}$$

for $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$\int_{|x-y|\leq 1} |u(y)|^p dy \rightarrow 0 \text{ as } |x| \rightarrow \infty,$$

this proves part (1). (2) Since V is in the Kato class K_N , this is a special case of Theorem C.4.2 of [5]. (3) This follows from Theorem C.3.4 of [5]. (4) By part (1), $u \in Y_\infty$ and, by parts (2) and (3), $u \in Y_1$. \square

We can now discuss the case in which $-\Delta + V : X_2 \rightarrow Y_2$ is an isomorphism.

Corollary 7. *Let the potential $V \in T_N(q)$ and suppose that $-\Delta + V : X_2 \rightarrow Y_2$ is an isomorphism. Then the bounded linear operator $S_p : X_p \rightarrow Y_p$ (defined by (4)) is also an isomorphism provided that $p \in (1, q] \cap (1, \infty)$.*

Proof. Fix a value of p according to the above restrictions. By Theorem 3, $S_p = -\Delta + V : X_p \rightarrow Y_p$ is a bounded linear operator and we shall show that

(a) $\ker S_p = \{0\}$, and (b) $\text{rge } S_p = Y_p$.

(a) Suppose that $u \in X_p$ and that $S_p u = 0$. Since $0 \notin \sigma$, the spectrum of S_2 , it follows from Proposition 6(4) that $u \in Y_s$ for all $s \in [1, \infty]$ and so, using Lemma 2, we see that $\{1 - V\}u \in Y_2$. Since $(-\Delta + 1)u = \{1 - V\}u \in Y_2$, Lemma 4(b) implies that $u \in X_2$ and so $u \in \ker S_2 = \{0\}$.

(b) Let $f \in Y_p$. We must show that there exists an element $u \in X_p$ such that $S_p u = f$. For this we consider a sequence $\{f_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $\|f_n - f\|_p \rightarrow 0$. Since $f_n \in Y_2$, there exists a unique element $u_n \in X_2$ such that $(-\Delta + V)u_n = f_n$. We now show that

(i) $u_n \in X_p$ and then that

(ii) $\{u_n\}$ is a Cauchy sequence in X_p .

(i) The inverse of the operator $-\Delta + V : X_2 \rightarrow Y_2$ is known (see [5], [6], [7]) to be an integral operator which we write as

$$Gf(x) = \int_{\mathbb{R}^N} g(x, y) f(y) dy$$

with a kernel g . Furthermore the estimates on the kernel imply that this integral operator acts as a bounded linear operator, $G_s : Y_s \rightarrow Y_s$, for any $s \in [1, \infty]$. Since

$$u_n(x) = \int_{\mathbb{R}^N} g(x, y) f_n(y) dy \quad \text{and} \quad f_n \in C_0^\infty(\mathbb{R}^N)$$

we can conclude that $u_n \in Y_s$ for all $s \in [1, \infty]$. In particular, $u_n \in L^\infty(\mathbb{R}^N)$ and, by Lemma 2, this ensures that $Qu_n \in Y_p$. But $u_n \in Y_p$ and so we also have that $(P-1)u_n \in Y_p$. Thus u_n is a tempered distribution satisfying the equation $(-\Delta + 1)u_n = h_n$ where $h_n = f_n - (V-1)u_n \in Y_p$. According to Lemma 4 this implies that $u_n \in X_p$.

(ii) By (i) we can now write $f_n = S_p u_n$. Since

$$[u_n - u_m](x) = \int_{\mathbb{R}^N} g(x, y)[f_n - f_m](y) dy$$

the boundedness of the operator $G_p : Y_p \rightarrow Y_p$ implies that $|u_n - u_m|_p \leq K(p) |f_n - f_m|_p$, which combined with Lemma 5 yields

$$\begin{aligned} \|u_n - u_m\|_{2,p} &\leq C(p) \left\{ |u_n - u_m|_p + |S_p[u_n - u_m]|_p \right\} \\ &\leq C(p) \left\{ K(p) |f_n - f_m|_p + |f_n - f_m|_p \right\} \end{aligned}$$

and so $\{u_n\}$ is indeed a Cauchy sequence in X_p .

There is an element $u \in X_p$ such that $\|u_n - u\|_{2,p} \rightarrow 0$ and so, by the continuity of $S_p : X_p \rightarrow Y_p$, this implies that $S_p u = f$. \square

Remark. The proof of Corollary 7 shows that, under the hypotheses of the corollary,

$$(S_p)^{-1} f(x) = \int_{\mathbb{R}^N} g(x, y) f(y) dy \text{ for all } f \in Y_p$$

where g is the kernel for the inverse of the operator $S_2 : X_2 \rightarrow Y_2$.

3. A perturbation lemma. In the previous section we dealt with the case where $-\Delta + V : X_2 \subset Y_2 \rightarrow Y_2$ is a self-adjoint isomorphism. In this section we establish a perturbation result which will be used to deal with points in its discrete spectrum. Although we shall only need this result in the case $S = -\Delta + V$, we give it for an arbitrary self-adjoint operator acting on $L^2(\mathbb{R}^N)$.

Lemma 8. *Let $S : D(S) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ be a self-adjoint operator such that $0 \notin \sigma_e(S)$. There exists a function $W \in C_0^\infty(\mathbb{R}^N)$ such that $W \geq 0$ and $\ker T = \{0\}$ where $T : D(T) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is the self-adjoint operator defined by $Tu = Su + Wu$ for $u \in D(T) = D(S)$.*

Remark. As the following proof shows, the function W can be chosen in such a way that its maximum is arbitrarily small and its support contains any given bounded subset of \mathbb{R}^N . We note that $S : D(S) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is a Fredholm operator since $0 \notin \sigma_e(S)$ and that $S - T : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is a bounded linear operator whose norm is less than or equal to $|W|_\infty$. Hence, by choosing $|W|_\infty$ small enough, we can ensure that $T : D(T) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is also a Fredholm operator with $\ker T = \{0\}$. This means that $T : D(T) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is in fact an isomorphism.

Proof. If $0 \notin \sigma(S)$, set $W \equiv 0$. Otherwise $0 \in \sigma_d(S)$ and we can choose an orthonormal basis $\{\varphi_i : 1 \leq i \leq m\}$ for $\ker S$ where $m < \infty$. Let $\xi \in C_0^\infty(\mathbb{R})$ be such that

$$\xi \geq 0 \quad \text{with} \quad \xi(t) = \begin{cases} 1 & \text{for } |t| \leq 1 \\ 0 & \text{for } |t| \geq 2 \end{cases}$$

and, for $R > 0$, set $W_R(x) = \xi(x/R)$. By dominated convergence it follows that

$$\int_{\mathbb{R}^N} W_R(x) \varphi_i(x) \varphi_j(x) dx \rightarrow \delta_{ij} \quad \text{as } R \rightarrow \infty.$$

Let

$$M = \left[\int_{\mathbb{R}^N} W_R(x) \varphi_i(x) \varphi_j(x) dx \right]$$

where R is chosen, and fixed, so that $\det M > 0$.

Given $w \in L^2(\mathbb{R}^N)$, let $\alpha(w) \in \mathbb{R}^m$ be the unique solution of the system

$$M\alpha = \begin{bmatrix} \int_{\mathbb{R}^N} W_R(x) \varphi_1(x) w(x) dx \\ \vdots \\ \int_{\mathbb{R}^N} W_R(x) \varphi_m(x) w(x) dx \end{bmatrix}.$$

There is a constant $K > 0$ such that

$$\left\{ \sum_{i=1}^m \alpha(w)_i^2 \right\}^{1/2} \leq K |w|_2.$$

Since $0 \notin \sigma_e(S)$, we have that $\delta \equiv \text{dist}(0, \sigma(S) \setminus \{0\}) > 0$. Thus we can choose $t > 0$ such that $t\sqrt{1 + K^2} < \delta$ and then set $W(x) = tW_R(x)$. Suppose

that $u \in D(S)$ with $(S + W)u = 0$. Then $u = v + w$ where $v \in \ker S$ and $w \in \{\ker S\}^\perp \cap D(S)$. Thus

$$v = \sum_{i=1}^m \alpha_i \varphi_i \quad \text{and} \quad Sw = -tW_R(v + w).$$

But $|Sz|_2 \geq \delta |z|_2$ for all $z \in \{\ker S\}^\perp \cap D(S)$ and so

$$\delta |w|_2 \leq t |W_R(v + w)|_2 \leq t |v + w|_2 \leq t \left\{ |v|_2^2 + |w|_2^2 \right\}^{1/2}$$

since $|W_R|_\infty = 1$. On the other hand, $-tP[W_R(v + w)] = PSw = 0$ where P is the orthogonal projection onto $\ker S$. This means that

$$\int_{\mathbb{R}^N} W_R(v + w) \varphi_i dx = 0 \quad \text{for } i = 1, \dots, m$$

and so

$$M \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = - \begin{bmatrix} \int_{\mathbb{R}^N} W_R(x) \varphi_1(x) w(x) dx \\ \vdots \\ \int_{\mathbb{R}^N} W_R(x) \varphi_m(x) w(x) dx \end{bmatrix}.$$

Thus $|v|_2 = \sqrt{\sum_{i=1}^m \alpha_i^2} \leq K |w|_2$ and consequently,

$$\delta |w|_2 \leq t \left\{ |v|_2^2 + |w|_2^2 \right\}^{1/2} \leq t \sqrt{1 + K^2} |w|_2.$$

By the choice of t , this implies that $w = 0$ which in turn implies that $v = 0$.

4. The main result. We are now ready to discuss the Fredholm properties of the operator $-\Delta + V$ in $L^p(\mathbb{R}^N)$.

Theorem 9. *Suppose that $V \in T_N(q)$ and consider the operator S_p defined by (4), where $p \in (1, q] \cap (1, \infty)$. Suppose that $\lambda \notin \sigma_e$ where σ_e is the essential spectrum of the self-adjoint operator $S_2 : X_2 \subset Y_2 \rightarrow Y_2$. Then*

- (i) $S_p - \lambda I : X_p \rightarrow Y_p$ is a Fredholm operator of index zero,
- (ii) $\ker(S_p - \lambda I) = \ker(S_2 - \lambda I)$ and
- (iii) $Y_p = \ker(S_p - \lambda I) \oplus \text{rge}(S_p - \lambda I)$.

Remarks. 1) The self-adjointness of $S_2 : X_2 \subset Y_2 \rightarrow Y_2$ has been established in Theorem 3.

2) This result contains Corollary 7 as a special case, since parts (i) and (ii) imply that $S_p - \lambda I : X_p \rightarrow Y_p$ is an isomorphism when $\lambda \notin \sigma$, the spectrum of $S_2 : X_2 \subset Y_2 \rightarrow Y_2$. Theorem 1 is obtained by setting $q = \infty$.

Proof. We fix $p \in (1, q] \cap (1, \infty)$. From Theorem 3 we know that $S_p - \lambda I : X_p \rightarrow Y_p$ is a bounded linear operator for all $p \in (1, q] \cap (1, \infty)$. Replacing V by $V - \lambda$, we can suppose that $\lambda = 0$. Thus $0 \notin \sigma_e$ and so $S_2 : X_2 \subset Y_2 \rightarrow Y_2$ is a Fredholm operator of index zero.

(i) Applying Lemma 8 to the self-adjoint operator $S = S_2 : X_2 \subset Y_2 \rightarrow Y_2$, we find that there is a function $W \in C_0^\infty(\mathbb{R}^N)$ such that $W \geq 0$ and $\ker(S_2 + W) = \{0\}$. Moreover, since $W \in C_0^\infty(\mathbb{R}^N)$, multiplication by W defines a compact operator from X_r into Y_r for all $r \in [1, \infty)$. (This follows immediately from the compactness of the embedding if $W^{1,r}(B)$ into $L^r(B)$ where B is a ball in \mathbb{R}^N containing the support of W .) This implies that the self-adjoint operator $S_2 + W : X_2 \subset Y_2 \rightarrow Y_2$ is also a Fredholm operator of index zero. Since $\ker(S_2 + W) = \{0\}$, this means that $S_2 + W : X_2 \subset Y_2 \rightarrow Y_2$ is in fact an isomorphism. Corollary 7 shows that $S_p + W : X_p \rightarrow Y_p$ is an isomorphism for all $p \in (1, q] \cap (1, \infty)$ since $V + W \in T_N(q)$. From the compactness of W as a multiplication operator from X_p to Y_p , it follows immediately that $S_p : X_p \rightarrow Y_p$ is a Fredholm operator of index zero.

(ii) Next we show that $\ker S_p = \ker S_2$. If $u \in \ker S_2$, it follows from Proposition 6(4) that $u \in Y_s$ for all $s \in [1, \infty]$. (Recall that, at the beginning of the proof, we have agreed that $0 \notin \sigma_e$.) Using Lemma 2 we see that $\{1 - V\}u \in Y_p$. Since $(-\Delta + 1)u = \{1 - V\}u \in Y_p$, Lemma 4(b) implies that $u \in X_p$ and so $u \in \ker S_p$.

Conversely if $u \in \ker S_p$, it follows from Proposition 6(4) that $u \in Y_s$ for all $s \in [1, \infty]$, and hence using Lemma 2, that $\{1 - V\}u \in Y_2$. Now Lemma 4(b) yields $u \in X_2$ and we have shown that $\ker S_p = \ker S_2$.

(iii) Finally we suppose that $u \in \ker S_p \cap \text{rge } S_p$. Thus, there exists $v \in X_p$ such that $S_p v = u$ where $u \in \ker S_p$. By part (ii) which has just been proved and Proposition 6, the latter property implies that $u \in \ker L_r$ for all $r \in (1, q] \cap (1, \infty)$ and $u \in Y_s$ for all $s \in [1, \infty]$.

We shall now show that $v \in X_T$ for some $T > \frac{N}{2}$. If $p > \frac{N}{2}$ this is clear. Suppose next that $1 < p < \frac{N}{2}$. Let $A = \{t \geq p : v \in X_t\}$. We know that $p \in A$ and we set $\alpha = \sup A$. Now if $v \in X_t$ for some $t \in [p, \frac{N}{2})$, it follows from (1) that $v \in Y_s$ provided that $t \leq s \leq t^*$, and hence by Lemma 2 that $Qv \in Y_T$

where $\frac{1}{T} = \frac{1}{q} + \frac{1}{t^*} = \frac{1}{q} + \frac{1}{t} - \frac{2}{N} < \frac{1}{t}$ since $q > \frac{N}{2}$. Furthermore, $(P-1)v \in Y_T$ since $T \leq t^*$ and $(P-1) \in Y_\infty$. Thus $u - (V-1)v = u - (P-1)v - Qv \in Y_T$ since $u \in Y_1 \cap Y_\infty$. But v satisfies the equation

$$(-\Delta + 1)v = u - (V-1)v$$

and so Lemma 4(b) now shows that $v \in X_T$ where $\frac{1}{T} - \frac{1}{t} = \frac{1}{q} - \frac{2}{N} < 0$. This proves that $\alpha > \frac{N}{2}$ and in fact $\alpha \geq q$.

The case where $p = \frac{N}{2}$ is similar but easier since we already know that $v \in Y_s$ for $s \in [p, \infty)$ in this case. Hence, by Lemma 2, $Qv \in Y_T$ for $1 < T < q$, and so $u - (V-1)v = u - (P-1)v - Qv \in Y_T$ for $T \in [p, q)$. Lemma 4(b) now shows that $v \in X_T$ for $T \in [p, q)$.

Thus, in all cases, $v \in X_T$ for some $T > \frac{N}{2}$ and so by (1) we now have that $v \in Y_\infty$. This implies that $Wv \in Y_s$ for all $s \in [1, \infty]$ since $W \in C_0^\infty(\mathbb{R}^N)$. Hence $u + Wv \in Y_s$ for all $s \in [1, \infty]$ and, in particular,

$$v \in X_p, \quad u + Wv \in Y_p \quad \text{and} \quad (-\Delta + V + W)v = u + Wv.$$

Thus by the remark at the end of Section 2,

$$v = \int_{\mathbb{R}^N} g(x, y) [u + Wv](x) dx$$

where g is the kernel for the self-adjoint isomorphism $-\Delta + V + W : X_2 \subset Y_2 \rightarrow Y_2$. But $u + Wv \in Y_2$ and so it follows from the properties of this integral mentioned earlier that $v \in Y_2$. This implies that $(P-1)v \in Y_2$, and, since $v \in Y_\infty$, Lemma 2 shows that $Qv \in Y_2$, too. But this means that

$$(-\Delta + 1)v = u - (V-1)v \in Y_2$$

and so we can conclude from Lemma 4(b) that $v \in X_2$. In other words, $u \in \ker S_2 \cap \text{rge } S_2$ which implies that $u = 0$ by the self-adjointness of S_2 . Thus $\ker S_p \cap \text{rge } S_p = \{0\}$ and since $S_p : X_p \rightarrow Y_p$ is a Fredholm operator of index zero we may conclude that $Y_p = \ker S_p \oplus \text{rge } S_p$. \square

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